Treatment of a conducting sphere in a constant electric field

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1 Solution to Laplace's equation in spherical coordinates

We assume that our constant electric field is along the negative z axis and therefore there will be symmetry about the ϕ axis. This will reduce our Laplace's equation to a two-dimensional case and make computation easier.

$$\nabla^2\Phi = \left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)\right\} + \left\{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)\right\} = 0 \tag{1}$$

Carrying out the partial differentiations and using the differential chain rule, we expand Laplace's equation.

$$\left\{\frac{\partial^2\Phi}{\partial r^2} + \frac{2}{r}\frac{\partial\Phi}{\partial r}\right\} + \left\{\frac{1}{r^2\tan\theta}\frac{\partial\Phi}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}\right\} = 0 \tag{2}$$

$$r\frac{\partial^2\Phi}{\partial r^2} + 2\frac{\partial\Phi}{\partial r} + \frac{1}{r\tan\theta}\frac{\partial\Phi}{\partial\theta} + \frac{1}{r}\frac{\partial^2\Phi}{\partial\theta^2} = 0 \tag{3}$$

We assume then perform separation of variables.

$$\Phi(r,\theta) = R(r)\Theta(\theta) \rightarrow R\Theta \tag{4}$$

$$r\frac{\partial^2 R\Theta}{\partial r^2} + 2\frac{\partial R\Theta}{\partial r} + \frac{1}{r\tan\theta}\frac{\partial R\Theta}{\partial\theta} + \frac{1}{r}\frac{\partial^2 R\Theta}{\partial\theta^2} = 0 \tag{5}$$

$$r\Theta R'' + 2\Theta R' + \frac{1}{r}R\Theta'' + \frac{1}{r\tan\theta}R\Theta' = 0 \tag{6}$$

Next, we divide through by $R\Theta$.

$$\frac{r^2R''}{R} + \frac{2rR'}{R} + \frac{\Theta''}{\Theta} + \frac{\Theta'}{\Theta\tan\theta} = 0 \tag{7}$$

If both radial and axial parts are to add up to zero, they can both be set to equal a positive and negative value of an arbitrary constant $m(m+1)$. This results in the following two differential equations.

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{2r}{R}\frac{dR}{dr} = m(m+1) \rightarrow r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - Rm(m+1) = 0 \tag{8}$$

$$\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} + \frac{1}{\Theta\tan\theta}\frac{d\Theta}{d\theta} = -m(m+1) \rightarrow \frac{d^2\Theta}{d\theta^2} + \frac{1}{\tan\theta}\frac{d\Theta}{d\theta} + \Theta m(m+1) = 0 \tag{9}$$

The solution to the radial and axial dependent components are then determined and combined to form a general solution for the electric potential Φ. The solution for the radial component is straight forward to verify, however, the solution for the axial component is a sum of Legendre's equations.

$$R(r) = Ar^m + \frac{B}{r^(m+1)} \tag{10}$$

$$\Theta(\theta) = \sum_m P_m(\cos\theta) \tag{11}$$

$$\Phi(r,\theta) = \sum_m P_m(\cos\theta)\Big(A_m r^m + \frac{B_m}{r^{(m+1)}}\Big) \tag{12}$$

2 Conductive sphere in a conductive medium in a constant electric field

For this section, we consider a sphere, with radius a, of conductivity σ_1 suspended in a medium of conductivity σ_2. It is assumed that the sphere is at the origin of the coordinate system and the applied electric field is along the negative z axis. We first determine the solution for the electric potential inside the sphere. Since the potential must be finite as r approaches zero, B_m is assumed to be zero for $r < a$.

$$\Phi_{in}(r,\theta) = \sum_m P_m(\cos\theta) A_m r^m : \qquad 0 < r < a \tag{13}$$

Outside of the sphere, we have the component of the solution that vanishes as r approaches ∞ as well as the applied electric field.

$$\Phi_{out}(r,\theta) = -E_0 r\cos\theta + \sum_m P_m(\cos\theta)\frac{B_m}{r^{(m+1)}} : \qquad a < r < \infty \tag{14}$$

Assuming that there is no interface charge, $\Phi_{in}(a,\theta)$ must equal $\Phi_{out}(a,\theta)$ so we can find a relationship between A_m and B_m coefficients for a given m index using equations 13 and 14. Only terms with the same θ dependence are equated, *i.e.* $P_1(\cos\theta) = \cos\theta$ so the static electric field is only included in the $m = 1$ boundary condition.

$$\begin{array}{rl} A_0 = B_0 a^{-1} & m = 0 \\ A_1 a = B_1 a^{-2} - E_0 a & m = 1 \\ A_2 a^2 = B_2 a^{-3} & m = 2 \\ \vdots & \\ A_m a^m = B_m a^{-(m+1)} & m > 2 \end{array} \tag{15}$$

We can now consider the boundary condition at the interface enforcing a continuous current density. Since we are working in spherical coordinates, the unit normal vector at the interface $\hat{n}$ is parallel to the unit radial vector $\hat{r}$.

$$\nabla \cdot \overrightarrow{J} = 0 \rightarrow \sigma_1 \frac{\partial \Phi_{in}}{\partial r}\Big\|_{r=a} = \sigma_2 \frac{\partial \Phi_{out}}{\partial r}\Big\|_{r=a} \tag{16}$$

$$\sigma_1 \frac{\partial}{\partial r}\Big(\sum_m P_m(\cos\theta) A_m r^m\Big) = \sigma_2 \frac{\partial}{\partial r}\Big(- E_0 r \cos\theta + \sum_m P_m(\cos\theta)\frac{B_m}{r^{(m+1)}}\Big) \tag{17}$$

$$\sigma_1 \sum_m P_m(\cos\theta) A_m \frac{\partial}{\partial r} r^m = -\sigma_2 E_0 r \cos\theta + \sigma_2 \sum_m P_m(\cos\theta)\frac{\partial}{\partial r} r^{-(m+1)} \tag{18}$$

It should be noted that the partial derivatives equation 18 should be evaluated for each m individually to evaluate a second relationship between the A_m and B_m coefficients.

$$\begin{aligned} 0 &= B_0 && m = 0 \\ \sigma_1 A_1 &= -2\sigma_2 B_1 a^{-3} - \sigma_2 E_0 && m = 1 \\ &\vdots \\ \sigma_1 m A_m a^{(}m-1) &= -(m+1)\sigma_2 B_m a^{-(m+2)} && m > 1 \end{aligned} \tag{19}$$

The only set of equalities that can hold in the non trivial case $(a \neq 0)$ from equations 15 and 19 are for $m = 1$. We can solve these linear equations by inverting a 2x2 matrix.

$$\begin{bmatrix} a & -a^{-2} \\ \sigma_1 & 2\sigma_2 a^{-3} \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} -E_0 a \\ -\sigma_2 E_0 \end{bmatrix} \tag{20}$$

$$\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} \frac{2\sigma_2 a^{-1}}{2\sigma_2 + \sigma_1} & \frac{1}{2\sigma_2 + \sigma_1} \\ \frac{-a^2 \sigma_1}{2\sigma_2 + \sigma_1} & \frac{a^3}{2\sigma_2 + \sigma_1} \end{bmatrix} \begin{bmatrix} -E_0 a \\ -\sigma_2 E_0 \end{bmatrix} \tag{21}$$

$$A_1 = \frac{-2\sigma_2 E_0}{2\sigma_2 + \sigma_1} + \frac{-\sigma_2 E_0}{2\sigma_2 + \sigma_1} = \frac{-3\sigma_2 E_0}{2\sigma_2 + \sigma_1} \tag{22}$$

$$B_1 = \frac{a^3 \sigma_1 E_0}{2\sigma_2 + \sigma_1} + \frac{-a^3 \sigma_2 E_0}{2\sigma_2 + \sigma_1} = \frac{a^3 E_0(\sigma_1 - \sigma_2)}{2\sigma_2 + \sigma_1} \tag{23}$$

Equations 22 and 23 can now be plugged into 13 and 14 to solve for the electric potential inside and outside of the conducting sphere.

$$P_1(\cos\theta) = \cos\theta \tag{24}$$

$$\Phi_{in}(r,\theta) = -E_0 r \cos\theta \frac{3\sigma_2}{2\sigma_2 + \sigma_1} : \qquad 0 < r < a \tag{25}$$

$$\Phi_{out}(r,\theta) = E_0 r \cos\theta \Big(\frac{a^3(\sigma_1 - \sigma_2)}{r^3(2\sigma_2 + \sigma_1)} - 1\Big) \qquad a < r < \infty \tag{26}$$